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On Covariant Phase Space and the Variational Bicomplex

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The notion of a phase space in classical mechanics is well known. The extension of this concept to field theory however, is a challenging endeavor, and over the years numerous proposals for such a generalization have appeared in the literature. In this paper We review a Hamiltonian formulation of Lagrangian field theory based on an extension to infinite dimensions of J.-M. Souriau's symplectic approach to mechanics. Following G. Zuckerman, we state our results in terms of the modern geometric theory of differential equations and the variational bicomplex. As an elementary example, we construct a phase space for the Monge–Ampere equation.

KEY WORDS: symplectic geometry; covariant phase space; space of motions; geometry of differential equations; variational bicomplex; Monge–Ampere equation.

1. INTRODUCTION

Bacry (1967) noted almost four decades ago that one can find the equations of motion of (spinning) elementary particles by studying Hamiltonian systems on coadjoint orbits of the Poincaré group. By doing so, he made the crucial observation that it is natural and important to introduce phase spaces not just as a set of p's and q's equipped with the canonical symplectic form $dq^i \wedge dp_i$, but as nontrivial symplectic manifolds (Crnković, 1987, 1988; Crnković and Witten, 1987). His work was put in a general context by J.-M. Souriau in his ground-breaking *Structure des Systemes Dynamiques* (Souriau, 1970). This seminal treatise is the first complete treatment of mechanics which fully utilizes the language and techniques of modern symplectic geometry.

It is now understood that a fruitful approach for treating dynamical problems with a finite number of degrees of freedom, is to model them as Hamiltonian systems on (in general nontrivial) symplectic manifolds (Abraham and Marsden, 1978; Arnold, 1989; Guillemin and Sternberg, 1984; Marsden, 1981; Marsden

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and Ratiu, 1999). It is less clear how to proceed when considering field theory. A rigorous approach based on (symplectic) manifolds modelled on Banach or Frechet spaces would perhaps be the best choice (Abraham and Marsden, 1978; Arnold, 1989; Marsden, 1981; Marsden and Ratiu, 1999) but to pursue such an endeavor is very delicate: results proven along these lines rely heavily on geometry *and* on hard nonlinear analysis, as J. Marsden's lecture notes (Marsden, 1981) testify.

A more formal approach to the Hamiltonian structure of (systems of) evolution partial differential equations is summarized in P. Olver's treatise (Olver, 1993) (see also Olver, 1989): One gives up the manifold description of phase space, and replaces the symplectic form by a "Hamiltonian differential operator." This point of view has been developed and applied with great success in integrable systems, giving rise, for instance, to the important structure of a "bi-hamiltonian system," which appears to encode the intuitive meaning of integrability for partial differential equations of evolutionary type.

If one is interested in the formal properties of evolution equations (e.g., conservation laws, symmetries, recursion operators (Nutku, 1996, 2001; Olver, 1989, 1993) it is natural to use this second point of view. On the other hand, if one is interested in canonical quantization of a dynamical system (Woodhouse, 1992) or in understanding bifurcations and the structure of the space of solutions (Marsden, 1981; Marsden and Ratiu, 1999) one most probably needs to possess a detailed understanding of the structure of the phase space of the system at hand, as in Marsden (1981) or Aldaya *et al.* (1992a,b,c), for example.

How are phase spaces constructed in field theory? The most common approach to field theory starts with a Lagrangian. One then uses a method due to Dirac (see (Abraham and Marsden, 1978; Cariñena and López, 1991; Crnković and Witten, 1987; Hanson *et al.*, 1976; Henneaux and Teitelboim, 1992) and references therein) to obtain a physical phase space equipped with canonical variables reminiscent of the p's and q's of classical mechanics. By doing so, one loses covariance, a fact usually seen as an imperfection from a physical point of view (Crnković, 1987, 1988; Crnković and Witten, 1987; Henneaux and Teitelboim, 1992). From a geometrical point of view, on the other hand the description of the phase space through canonical coordinates appears incomplete: One would like to present it in an intrinsic, global fashion.

A way to repair these shortcomings, and to move from a Lagrangian point of view to a Hamiltonian picture, is to stay "in between" the formal versions of the Hamiltonian formalism (Hanson *et al.*, 1976; Olver, 1989, 1993) and its completely rigorous symplectic version (Abraham and Marsden, 1978; Arnold, 1989; Marsden, 1981; Marsden and Ratiu, 1999): One may attempt to obtain a covariant, coordinate-free description of the phase space, as a first step towards finding a description of the dynamics *à la Marsden* (Marsden, 1981) say, and also as a previous step to canonical quantization (Woodhouse, 1992). This has been accomplished by G. Zuckerman—using formal arguments rooted in the rigorous theory of the variational bicomplex—in a beautiful and not so well-known paper (Zuckerman, 1987) written in 1986. He appears to have been the first in explaining, in full generality, how to build phase spaces for Lagrangian theories with finite or infinite number of degrees of freedom in a covariant way, using directly the Lagrangian and without going through Dirac's theory of constraints.

Historically, some special cases of Zuckerman's analysis appeared before (Zuckerman, 1987), notably in Souriau's treatise (Souriau, 1970), in N. Woodhouse's monograph on geometric quantization (Woodhouse, 1992), in S. Sternberg's analysis of the formal variational calculus of Gel'fand and Dikii (Sternberg, 1978), and in fact in the work of J. L. Lagrange himself, as pointed out in Landi and Rovelli (1997, 1998). Related and subsequent work on the subject are, among others, the inspiring paper written by Crnković and Witten (1987) on the covariant description of phase spaces for Yang–Mills theory and general relativity, the Hamiltonian analysis by Crnković (1987, 1988) of a general first-order Lagrangian (super)theory and superstrings, the covariant phase space analysis of two-dimensional gravity models by Navarro-Salas, Navarro and Aldaya (Aldaya *et al.*, 1992a,b,c), and the proposal by Landi and Rovelli (1997, 1998) of Dirac eigenvalues as observables in euclidean gravity.

In spite of these advances however, no complete exposition of Zuckerman's ideas seems to be available in the literature, except for a review of Zuckerman (1987), appearing in a recent abstract exposition of field theory (Deligne and Freed, 1999). Due to the importance of Zuckerman (1987), it is natural to try to fill this gap. In this paper we consider some aspects of Zuckerman's article from the point of view of the geometric theory of differential equations and the variational bicomplex (Anderson, 1989, 1992; Anderson and Kamran, 1995; Olver, 1989, 1993), taking advantage of the fact that the great development of these subjects allows us to be both precise and concrete. This work is organized as follows: Section 2 is a summary of some relevant facts on symplectic and presymplectic manifolds based mainly in Abraham and Marsden, 1978; Marsden and Ratiu (1999) and Souriau (1970). The variational bicomplex is studied in Section 3, and Zuckerman's construction is presented in Section 4. As an example, we apply Zuckerman's ideas to the Monge–Ampere equation in Section 5. A fuller exposition of these and related matters will appear elsewhere.

The Einstein summation convention will be used throughout.

2. HAMILTONIAN SYSTEMS AND PRESYMPLECTIC MANIFOLDS

We review the relation between Hamiltonian mechanics and *Souriau reduction*, that is, the understanding of the equations of motion as a (perhaps local) description of the leaves of a foliation of a presymplectic manifold (Souriau, 1970). The manifolds appearing in this section are all finite-dimensional. All maps, vector fields and tensors are assumed to be of class C^∞.

2.1. Presymplectic and Symplectic Manifolds

A two-form ω on a manifold M is a *presymplectic form* on M if it is closed and of constant rank on M. If, in addition, ω is non degenerate (that is, if the rank of ω is equal to the dimension of M) we say that (M, ω) is a *symplectic manifold* and that ω is a *symplectic form* on M. From now on, the adjective "presymplectic" will be applied exclusively to closed two-forms of constant rank strictly less than $\dim(M)$.

A standard example of symplectic manifold is the cotangent bundle T^*M of a given manifold M (Abraham and Marsden, 1978; Arnold, 1989; Marsden and Ratiu, 1999). In coordinates, if (q^i) is a coordinate chart on M, and $\alpha_q = (q^1, \ldots, q^n, p_1, \ldots, p_n)$ is an element of T^*M, then $\omega_0 = dq^i \wedge dp_i$ is a symplectic form on T^*M. The symplectic manifold (T^*M, ω_0) is called the *canonical phase space* of the *configuration space M*.

The local characterization of (pre)symplectic forms is given by Darboux theorem (Abraham and Marsden, 1978).

Theorem 2.1. *Suppose that ω is a non degenerate two-form on a $2n$-dimensional manifold M. Then $\mathrm{d}\omega = 0$ if and only if for any $m \in M$ there exists a chart (U, ϕ) about m such that $\phi(m) = 0$ and*

$$\omega|_U = dx^i \wedge dy_i, \tag{1}$$

in which $\phi|_U = (x^1, \ldots, x^n, y_1, \ldots, y_n)$. More generally, if (M, ω) is a $(2n + k)$–dimensional presymplectic manifold with $\mathrm{rank}(\omega) = 2n$, for each $m \in M$ there is a chart (U, ψ) about m such that

$$\omega|_{\mathrm{U}} = dq^i \wedge dr_i \ ,$$

in which $\psi|_U = (q^1, \ldots, q^n, r_1, \ldots, r_n, w^1, \ldots, w^k)$.

2.2. Hamiltonian Systems

Let (M, ω) be a symplectic manifold and let $H : M \to \mathbb{R}$ be a smooth function on M. The triplet (M, ω, H) is called the *Hamiltonian system on* (M, ω) with *Hamiltonian function H* and *phase space* (M, ω). The *evolution* of the system (M, ω, H) is given by the flow of the *Hamiltonian vector field X_H* uniquely determined by the equation

$$i_{X_H}\omega = dH. \tag{2}$$

(The one-form $i_{X_H}\omega$ on M is given by $i_{X_H}\omega(Y) = \omega(X_H, Y)$ for any vector field Y on M.) That Eq. (2) does encode Hamilton's equations is a consequence of Darboux's result:

Proposition 2.1. *Let (M, ω) be a symplectic manifold and suppose that $(q^1, \ldots, q^n, p_1, \ldots, p_n)$ are canonical coordinates (i.e. given by Darboux's theorem) on M, and let $H : M \to \mathbb{R}$ be a smooth function on M. Then, the equation $i_{X_H}\omega = dH$ implies that $X_H = (\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q^i})$. Thus $(q(t), p(t))$ is an integral curve of X_H if and only if*

$$\frac{dq^i}{dt} = \frac{\partial H}{\partial p_i}, \qquad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q^i}, \quad i = 1, \ldots, n. \tag{3}$$

2.3. The Space of Motions

It is not always straightforward to find a symplectic description of a mechanical system (Abraham and Marsden, 1978; Guillemin and Sternberg, 1984; Marsden, 1981; Souriau, 1970): as stated in the Introduction, it is not uncommon to consider systems described by a (singular) Lagrangian (Cariñena and López, 1991; Hanson *et al.*, 1976; Henneaux and Teitelboim, 1992), and to find canonical formulations for them by means of the Dirac constraint algorithm (Abraham and Marsden, 1978; Cariñena and López, 1991; Crnković and Witten, 1987; Hanson *et al.*, 1976; Henneaux and Teitelboim, 1992; Nutku, 1996, 2001; Woodhouse, 1992). When applied to systems with a finite number of degrees of freedom, the final result of this algorithm is in general a presymplectic manifold (M, ω). Now, given a dynamical system described by a presymplectic manifold (M, ω)—which was obtained perhaps by means of the Dirac constraint algorithm—the corresponding phase space is constructed as follows (Souriau, 1970):

For each $v \in M$ we set $\ker_v\omega = \{Z_v \in T_vM : i_{Z_v}\omega = 0\}$, and define the distribution of vector spaces

$$\ker\omega = \bigcup_{v\in M} \ker_v\omega. \tag{4}$$

Since ω is of constant rank, the dimension of $\ker_v\omega$ is independent of v. Moreover if Z and Y are vector fields on M such that $Z(v)$ and $Y(v)$ belong to $\ker_v\omega$ for all $v \in M$, then, $i_{[Z,Y]}\omega = L_Z(i_Y\omega) - i_Y(L_Z\omega) = 0 - i_Y\ (d(i_Z\omega) + i_Z d\omega) = 0$, in which for any differential form σ, $L_Z\sigma$ is the Lie derivative of σ along Z, see (Abraham and Marsden, 1978), and so $[Z, Y](v) \in \ker_v\omega$ for all $v \in M$. Frobenius' theorem (Abraham and Marsden, 1978; Marsden and Ratiu, 1999) implies that the distribution $\ker\omega$ is integrable, that is, there exists a foliation $\Phi_\omega = \{\mathcal{L}_\alpha\}_{\alpha\in A}$ of M such that its tangent bundle $T(M, \Phi_\omega)$, in which

$$T(M, \Phi_\omega) = \bigcup_{\alpha\in A} \bigcup_{v\in\mathcal{L}_\alpha} T_v\mathcal{L}_\alpha,$$

is equal to $\ker\omega$.

Definition 2.1. If a dynamical system is described by a presymplectic manifold (M, ω), we call (M, ω) the evolution space of the system.

If (M, ω) is an evolution space, the space of motions of (M, ω) is $U_M = M/\ker \omega$, that is, U_M is the set of leaves of the foliation Φ_ω determined by the integrable distribution (4).

Given a presymplectic manifold (M, ω), the procedure of constructing the corresponding space of motions U_M will be referred to as *Souriau reduction*, after the fundamental contributions to the subject made by Souriau (1970).

An useful criterion for assuring that U_M is a manifold is the following (see (Marsden, 1981; Marsden and Ratiu, 1999; Woodhouse, 1992) and references therein): U_M is a manifold if and only if for every $v \in M$ there exists a local submanifold Σ_v of M such that Σ_v intersects every leaf of the foliation Φ_ω in at most one point (or nowhere), and $T_v\Sigma_v \oplus T_v(M, \Phi_\omega) = T_v M$. The submanifold Σ_v is called a *slice* or *local cross section* for Φ_ω. It follows that if U_M is a manifold, its dimension is equal to $\dim(M) - \dim(\ker \omega)$.

The next theorem, which goes back at least to Souriau (1970; see Abraham and Marsden, 1978; Arnold, 1989; Guillemin and Sternberg, 1984) is the main result on this subject:

Theorem 2.2. *Let (M, ω) be a presymplectic manifold and assume that the space of motions U_M is a manifold. Then, U_M can be equipped with a symplectic structure $\widetilde{\omega}$ such that $\pi^*\widetilde{\omega} = \omega$, in which $\pi : M \to M/\ker \omega$ is the canonical projection from M onto U_M.*

The true phase space for a mechanical system modelled on a presymplectic manifold (M, ω) is the symplectic manifold $(U_M, \widetilde{\omega})$ constructed in the last theorem. A very interesting application of this point of view is Künzle's (1972) discovery of a genuine presymplectic description of a spinning particle in an external gravitational field: this paper appears to be the first deep physical application of Souriau reduction. Later, S. Sternberg and his coworkers formulated a program to reduce classical mechanics to the construction of presymplectic manifolds and the corresponding spaces of motion. Their work is summarized in (Guillemin and Sternberg, 1984).

We now explain the name "space of motions." Souriau's original discussion on the connection between spaces of motions and Hamiltonian systems is in Souriau (1970, pp. 128–132). We begin with two elementary lemmas:

Lemma 2.3. *Let (M, ω, H) be a Hamiltonian system on the symplectic manifold (M, ω). Define $N = M \times \mathbb{R}$, and set $\Omega = p_1^*\omega + (p_1^*dH) \wedge (p_2^*dt)$, in which $p_1 : N \to M$ and $p_2 : N \to \mathbb{R}$ are the canonical projection maps. Then, (N, Ω) is a presymplectic manifold.*

The proof of the lemma consists in checking that for each $(m, t) \in N$, $\ker_{(m,t)} \Omega = \{(\alpha X_H(m), \alpha) : \alpha \in \mathbb{R}\}$, so that the dimension of $\ker_{(m,t)}\Omega$ is equal to 1 for all $(m, t) \in N$. Of course, if $m(t)$ is an integral curve of X_H and $n(t) = (m(t), t)$, then $n'(t) \in \ker_{n(t)}\Omega$ for all t. Conversely we recover, up to parameterizations, the integral curves of X_H:

Lemma 2.4. *Let (M, ω, H) be a Hamiltonian system on the symplectic manifold (M, ω), and let (N, Ω) be the presymplectic manifold defined in Lemma 2.3. The integral curves of the Hamiltonian system (M, ω, H) can be obtained, up to parametrization, by projecting the leaves of the foliation Φ_Ω into M.*

Proof: Parametrize a leaf $\mathcal{L} \in \Phi_\Omega$ by means of a curve $n(s) = (m(s), \gamma(s))$ satisfying $n'(s) \in \ker_{n(s)}\Omega$ and $n'(s) \neq 0$ for all s. Then $n(s)$ can be reparametrized to be of the form $(m(t), t)$, where $m(t)$ is an integral curve of X_H: since $n'(s) = (m'(s), \gamma'(s)) \in \ker_{n(s)}\Omega$ then $m'(s) = \gamma'(s)X_H(m(s))$, and $\gamma'(s) \neq 0$ for all s; thus, if one sets $t = \gamma(s)$, then $n(t) = ((m \circ \gamma^{-1})(t), t)$, and $n'(t) = (X_H(m \circ \gamma^{-1}(t)), 1)$. □

Following Souriau (1970), we then *identify the motions of the system described by (M, ω, H) with the leaves of the foliation induced by the integrable distribution ker Ω*. We obtain the following result:

Proposition 2.2. *Define N and Ω as in Lemma 2.3, and identify the leaves of the foliation Φ_Ω with the integral curves of X_H. Then, the space of motions $U_N = N/\ker\Omega$ is a manifold. Moreover, the symplectic manifolds (M, ω) and $(U_N, \widetilde{\Omega})$, in which $\widetilde{\Omega}$ is the symplectic form on U_N determined by Theorem 2, are symplectomorphic, that is, there exists a diffeomorphism $\lambda : U_N \to M$ such that $\lambda^*\omega = \widetilde{\Omega}$.*

Proof: The idea of the proof is that, as explained above, a leaf $\mathcal{L} \in U_N$ can be described by a curve $(m(t), t)$, in which $m(t)$ is an integral curve of X_H. We can then define the map $\lambda : U_N \to M$ by $\lambda(\mathcal{L}) = m(0)$.

More rigorously, we use coordinates: the flow box theorem (Abraham and Marsden, 1978) says that for each $m \in M$ there exists an open set $U_m \subseteq M$ and a smooth map $F : U_m \times I \to M$, in which $I = (-a, a)$ with $a > 0$ or $a = \infty$, such that for each $v \in U_m$, the curve $c_v : I \to M$ given by $c_v(s) = F(v, s)$ is the integral curve of X_H passing through v. Now, since the leaves of N through $u \in U_m$ are precisely the integral curves of X_H, the submanifold $\Sigma = U_m \times \{0\}$ is a slice for the foliation Φ_Ω. Thus, U_N is a manifold, and for $(u, 0) \in \Sigma$, we simply define the function λ as the projection $\lambda(u, 0) = u$. This is of course a bijective smooth symplectic map. □

In conclusion, *we have identified the phase space of a dynamical system with the space of classical solutions of the system at hand*. This identification is at the core of the generalization of Souriau's point of view to Lagrangian field theory (Crnković and Witten, 1987; Woodhouse, 1992; Zuckerman, 1987): in this infinite dimensional context, one considers the space of all classical solutions to the equations of motion, and equips it with a presymplectic structure. One then constructs the corresponding "space of motions," the genuine phase space of the theory. We review this generalization in Section 4. Since we will state our results in terms of the variational bicomplex, we discuss this important tool first.

3. THE VARIATIONAL BICOMPLEX

In this section we quickly review the modern geometrical setting for differential equations and introduce the variational bicomplex. Our main references for these matters are (Anderson, 1989, 1992; Anderson and Kamran, 1995) and Olver (1993).

3.1. Geometry of Infinite Jets

Let $\pi : E \to M$ be a trivial fiber bundle. The manifold M is the space of independent variables x^i, $1 \leq i \leq n$, and the typical fiber is the space of the dependent variables u^α, $1 \leq \alpha \leq m$. We also let $J^k E$, $k \geq 1$, be the bundle of k-jets of local sections of E.

The *infinite jet bundle* $J^\infty E \to M$ is the inverse limit of the tower of jet bundles $M \leftarrow E \cdots \leftarrow J^k E \leftarrow J^{k+1} E \leftarrow \cdots$ under the standard projections $\pi^k_l : J^k E \to J^l E, k > l$. Locally, $J^\infty E$ is described by canonical coordinates $(x^i, u^\alpha, u^\alpha_{i_1}, \ldots, u^\alpha_{i_1 i_2 \ldots i_k}, \ldots)$, $1 \leq i_1 \leq i_2 \leq \cdots \leq i_k \leq n$, obtained from the standard coordinates on the finite-order jet bundles $J^k E$,

$$u^\alpha_{i_1}(j^k(s)(p)) = \frac{\partial s^\alpha}{\partial x^{i_1}}(p), \qquad u^\alpha_{i_1 i_2}(j^k(s)(p)) = \frac{\partial^2 s^\alpha}{\partial x^{i_1} \partial x^{i_2}}(p)\,, \qquad \ldots\,, \tag{5}$$

in which $p \in M$ and $j^k(s)$ is the k–jet of the local section $s : (x^i) \mapsto (x^i, s^\alpha(x^i))$ of E.

Any local section $s : (x^i) \mapsto (x^i, s^\alpha(x^i))$ of E lifts to a unique local section $j^\infty(s)$ of $J^\infty E$ called the *infinite prolongation* of s. In coordinates, $j^\infty(s)$ is the section

$$\left(x^i, s^\alpha(x^i), \frac{\partial s^\alpha}{\partial x^{i_1}}(x^i), \ldots, \frac{\partial^k s^\alpha}{\partial x^{i_1} \cdots \partial x^{i_k}}(x^i), \ldots \right).$$

A function $f : J^\infty E \to \mathbb{R}$ is *smooth* if it factors through a finite-order jet bundle, that is, if $f = f_k \circ \pi^\infty_k$ for some function $f_k : J^k E \to \mathbb{R}$, in which $\pi^\infty_k : J^\infty E \to J^k E$ denotes the canonical projection from $J^\infty E$ onto $J^k E$.

A *vector field* X on $J^\infty E$ is a derivation on the ring of smooth functions on $J^\infty E$. In local coordinates, vector fields are formal series of the form

$$X = A_i \frac{\partial}{\partial x^i} + \sum_{\substack{k\geq 0 \\ 1\leq i_1\leq\cdots\leq i_k\leq n}} B^\alpha_{i_1\ldots i_k} \frac{\partial}{\partial u^\alpha_{i_1\ldots i_k}}, \tag{6}$$

in which A_i, $B^\alpha_{i_1\ldots i_k}$ are smooth functions on $J^\infty E$. Vector fields X on M can be canonically prolonged to vector fields $pr^\infty X$ on $J^\infty E$: the action of the derivation $pr^\infty X$ on a smooth function f on $J^\infty E$ is given by

$$pr^\infty X\,(j^\infty(s)(p)) \cdot f = X(p) \cdot (f \circ j^\infty(s)) \tag{7}$$

for all local sections $j^\infty(s)$ of $J^\infty E$ and all $p \in M$. This prolongation operation defines a connection $\mathcal{C}$ on $J^\infty E$ called the *Cartan connection*: the horizontal lift of a vector field X on M, also called the *total derivative in the X direction*, is simply $pr^\infty X$. Locally, horizontal vector fields are linear combinations of the total derivatives D_j, in which $D_j = pr^\infty(\partial/\partial x^j)$, that is,

$$D_j = \frac{\partial}{\partial x^j} + u^\alpha_j \frac{\partial}{\partial u^\alpha} + u_{i_1 j} \frac{\partial}{\partial u^\alpha_{i_1}} + u_{i_1 i_2 j} \frac{\partial}{\partial u^\alpha_{i_1 i_2}} + \ldots. \tag{8}$$

The prolongation operation (7) satisfies $pr^\infty[X_1, X_2] = [pr^\infty X_1, pr^\infty X_2]$ for all vector fields X_1 and X_2 on M, and therefore the Cartan connection is flat.

Differential forms on $J^\infty E$ are the pull-backs of differential forms on $J^k E$ by the projections π^∞_k. Thus, any differential k-form ω on $J^\infty E$ may be written in canonical coordinates as a finite linear combination of terms

$$A\, dx^{i_1} \wedge \cdots \wedge dx^{i_p} \wedge du^{\alpha_1}_{j_1\cdots j_{p_1}} \wedge \ldots \wedge du^{\alpha_q}_{l_1\cdots l_{p_q}}, \tag{9}$$

in which $p + q = k$ and A is a smooth function on $J^\infty E$. A differential form ω on $J^\infty E$ is called a *contact form* if $j^\infty(s)^*\omega = 0$ for all local sections s of E. The set of contact forms determines an ideal $\mathcal{I}$ of the ring of differential forms on $J^\infty E$. Locally, the *contact ideal* $\mathcal{I}$ is generated by the *basic contact one-forms*

$$\theta^\alpha_{i_1\ldots i_k} = du^\alpha_{i_1\ldots i_k} - u^\alpha_{i_1\ldots i_k j}\, dx^j, \qquad k \geq 0, \tag{10}$$

and it is not hard to check that the exterior derivative of $\theta^\alpha_{i_1\ldots i_k}$ is given by

$$d\theta^\alpha_{i_1\ldots i_k} = dx^j \wedge \theta^\alpha_{i_1\ldots i_k j}. \tag{11}$$

Contact forms are important because they provide a dual description of the Cartan connection: a vector field X on $J^\infty E$ is horizontal if and only if $i_X\omega = 0$ for all one-forms $\omega \in \mathcal{I}$.

3.2. The Variational Bicomplex

To define the variational bicomplex we bigrade the differential forms on $J^\infty E$ using the Cartan connection of $J^\infty E$ (Anderson, 1989, 1992; Anderson and Kamran, 1995): A p-form on $J^\infty E$ is *of type* (r, s), in which $r + s = p$, if $\omega(X_1, \ldots X_p) = 0$ whenever either

(a) more than s of the vector fields $X_1, \ldots X_p$ are π^∞_M-vertical (that is, if we write $X_1, \ldots X_p$ in local coordinates as in (6), all the coefficients A_i vanish) or
(b) more than r of the vector fields $X_1, \ldots X_p$ are horizontal with respect to the connection $\mathcal{C}$ of $J^\infty E$.

In coordinates, a p-form ω is of type (r, s) if it can be written as a finite sum of terms of the form

$$A\, dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge \theta^{\alpha_1}_{j_1\cdots j_{p_1}} \wedge \ldots \wedge \theta^{\alpha_s}_{l_1\cdots l_{p_s}}. \tag{12}$$

Now, let $\Omega^p(J^\infty E)$ denote the set of p-forms on $J^\infty E$, and let $\Omega^{r,s}(J^\infty E)$ denote the set of p-forms of type (r, s). Then, we have the direct sum decomposition

$$\Omega^p(J^\infty E) = \bigoplus_{r+s=p} \Omega^{r,s}(J^\infty E).$$

Equation (11) implies that the exterior derivative d on $J^\infty E$ splits,

$$d : \Omega^{r,s}(J^\infty E) \to \Omega^{r+1,s}(J^\infty E) \oplus \Omega^{r,s+1}(J^\infty E),$$

and we can write $d = d_H + d_V$, in which $d_H : \Omega^{r,s}(J^\infty E) \to \Omega^{r+1,s}(J^\infty E)$, and $d_V : \Omega^{r,s}(J^\infty E) \to \Omega^{r,s+1}(J^\infty E)$ are the *horizontal* and *vertical* exterior derivatives respectively. The equation $d^2 = 0$ implies that $d_H^2 = d_V^2 = 0$ and $d_H d_V + d_V d_H = 0$. In local coordinates, d_H and d_V are computed by means of the following formulae:

$$d_H f = \sum_i (D_{x^i} f) dx^i; \tag{13}$$

$$d_V f = \frac{\partial f}{\partial u^\alpha}\theta^\alpha + \frac{\partial f}{\partial u^\alpha_i}\theta^\alpha_i + \frac{\partial f}{\partial u^\alpha_{ij}}\theta^\alpha_{ij} + \ldots; \tag{14}$$

$$d_H(dx^i) = 0, \quad d_H\theta^\alpha_{i_1\ldots i_k} = \sum_j dx^j \wedge \theta^\alpha_{i_1\ldots i_k j}; \tag{15}$$

$$d_V(dx^i) = 0, \quad d_V\theta^\alpha_{i_1\ldots i_k} = 0. \tag{16}$$

Thus, for example, $d_V x^i = 0$ and $d_V u^\alpha_{i_1\ldots i_k} = \theta^\alpha_{i_1\ldots i_k}$.

The *variational bicomplex* for the bundle E is the double complex $(\Omega^{*,*}(J^\infty E), d_H, d_V)$ of differential forms on the infinite jet bundle $J^\infty E$. In detail, writing $\Omega^{*,*}$ for $\Omega^{*,*}(J^\infty E)$, this important bicomplex looks like follows:

$$
\begin{array}{ccccccccccccc}
 & & & & \vdots & & & & & & & & \vdots \\
 & & & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V \\
 & & 0 & \longrightarrow & \Omega^{0,2} & \xrightarrow{d_H} & \Omega^{1,2} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{n-1,2} & \xrightarrow{d_H} & \Omega^{n,2} \\
 & & & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V \\
 & & 0 & \longrightarrow & \Omega^{0,1} & \xrightarrow{d_H} & \Omega^{1,1} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{n-1,1} & \xrightarrow{d_H} & \Omega^{n,1} \\
 & & & & \uparrow d_V & & \uparrow d_V & & & & \uparrow d_V & & \uparrow d_V \\
0 & \longrightarrow & \mathbb{R} & \longrightarrow & \Omega^{0,0} & \xrightarrow{d_H} & \Omega^{1,0} & \xrightarrow{d_H} & \dots & \xrightarrow{d_H} & \Omega^{n-1,0} & \xrightarrow{d_H} & \Omega^{n,0}
\end{array}
\tag{17}
$$

If the fiber bundle $E \to M$ is simply $\mathbb{R}^{m+n} \to \mathbb{R}^n$, *all the sequences appearing in* (17)*, both horizontal and vertical, are exact*. This important result has been proven by several researchers, notably I. M. Anderson, L. Dickey, F. Takens, W. M. Tulczyjew, T. Tsujishita, and A. M. Vinogradov. Original references appear in Anderson (1989, 1992), Anderson and Kamran (1995), Olver (1993), and Zuckerman (1987).

4. HAMILTONIAN FORMALISM FOR LAGRANGIAN FIELD THEORIES

We fix a fiber bundle $\pi : E \to M$ as in Section 3 and let $J^\infty E$ be the infinite jet bundle of E. The space $\Omega^{n,1}(J^\infty E)$ possesses a distinguished subspace $\mathcal{E}^{n+1}(E)$ of all *source forms* on $J^\infty E$: we say that a differential form $\omega \in \Omega^{n,1}(J^\infty E)$ is a *source form* if the local representative of ω in any system of coordinates (x^i, u^α) on E can be written as

$$\omega = P_\beta\big(x^i, u^\alpha, u^\alpha_i, \dots, u^\alpha_{i_1 \dots i_k}\big)\, du^\beta \wedge dx^1 \wedge \dots \wedge dx^n \ ,$$

or, equivalently, if $\omega = P_\beta(x^i, u^\alpha, u^\alpha_i, \dots, u^\alpha_{i_1 \dots i_k})\,\theta^\beta \wedge dx^1 \wedge \dots \wedge dx^n$. An intrinsic characterization of the space of source forms is in Anderson (1989, 1992). Their importance is due to the following lemma (Anderson, 1989, 1992; Zuckerman, 1987).

Lemma 4.5. *Assume that $\omega \in \Omega^{n,1}(J^\infty E)$. Then, ω can be written uniquely as*

$$\omega = \omega_1 + d_H \eta \ , \tag{18}$$

in which $\omega_1 \in \mathcal{E}^{n+1}(E)$ is a source form and $\eta \in \Omega^{n-1,1}(J^\infty E)$.

Suppose now that we fix a Lagrangian density $\lambda \in \Omega^{n,0}(J^\infty E)$, so that in local coordinates (x^i, u^α), $\lambda = L(x^i, u^\alpha, u^\alpha_i, \dots, u^\alpha_{i_1 \dots i_k}) dx^1 \wedge \dots \wedge dx^n$, (see Anderson, 1989, 1992; Olver, 1993; Zuckerman, 1987). The vertical exterior derivative $d_V \lambda$ belongs to $\Omega^{n,1}(J^\infty E)$, and the last lemma implies that we can write

$$d_V\lambda = E(\lambda) + d_H\eta \tag{19}$$

uniquely, in which $E(\lambda)$ is a source form and $\eta \in \Omega^{n-1,1}(J^\infty E)$. We remark that essentially, $E(\lambda)$ is the Euler–Lagrange operator evaluated at L, that is, $E(\lambda) = E_\alpha(L)du^\alpha \wedge dx^1 \wedge \cdots \wedge dx^n$, in which

$$E_\alpha(L) = \frac{\partial L}{\partial u^\alpha} - D_i\left(\frac{\partial L}{\partial u^\alpha_i}\right) + D_iD_j\left(\frac{\partial L}{\partial u^\alpha_{ij}}\right) - \cdots,$$

(see Anderson, 1989, 1992; Olver, 1993). We now define $U(\lambda) \in \Omega^{n-1,2}(J^\infty E)$ by

$$U(\lambda) = d_V\eta\ , \tag{20}$$

in which η is determined by the fundamental equation (19). The differential form $U(\lambda)$ is *Zuckerman's universal current*. We observe that $d_V U(\lambda) = 0$ and that, less trivially, the horizontal derivative $d_H U(\lambda)$ *vanishes on solutions* $u^\alpha(x^i)$ *to the Euler–Lagrange equations*. Indeed, *on solutions to* $E_\alpha(L) = 0$, Eq. (19) becomes $d_V\lambda = d_H\eta$, and therefore

$$0 = d_Vd_V\lambda = d_Vd_H\eta = -d_Hd_V\eta = -d_HU(\lambda)\ . \tag{21}$$

After Zuckerman (1987), we say that the differential $(n-1,2)$-form $U(\lambda)$ is a *conserved current* for the Euler–Lagrange equations $E_\alpha(L) = 0$, in the sense of the following definition:

Definition 4.2. Fix a form $\lambda \in \Omega^{n,0}(J^\infty E)$ as above. A differential form $K \in \Omega^{n-1,q}(J^\infty E)$, $q \geq 0$, is a conserved current for the Euler–Lagrange equations $E_\alpha(L) = 0$ if

$$d_H K = 0$$

whenever $u^\alpha(x^i)$ is a solution to the equations $E_\alpha(L) = 0$.

The conserved currents of Definition 4.2 generalize the standard conservation laws of field theory: as is well-known, conservation laws are usually defined as differential forms $K \in \Omega^{n-1,0}(J^\infty E)$ which are closed on solutions, (see Anderson, 1989, 1992; Anderson and Kamran, 1995; Abraham and Marsden, 1978; Olver, 1989, and references therein). The importance of these new conserved currents, also called *higher-degree* or *form-valued conservation laws*, has been recognized only recently (Anderson, 1989, 1992; Anderson and Kamran, 1995).

Definition 4.2 extends to arbitrary systems of partial differential equations. Thus, for example, it is a straightforward exercise to check that the canonical symplectic form $\omega_0 = dq^i \wedge dp_i$ considered in Section 2 is a (0,2)-differential form on $J^\infty E$ (with $E \to M$ being the trivial bundle $\mathbb{R}^{2n+1} \to \mathbb{R}$ with coordinates

(t, q^i, p_i), $i = 1, \ldots n$) which is a form-valued conservation law for Hamilton's equations.

The following definition replaces the finite dimensional manifolds of Section 2:

Definition 4.3.

(a) The solution variety S_L associated with a Lagrangian density $\lambda = L(x^i, u^\alpha, u_i^\alpha, \ldots, u^\alpha_{i_1 \ldots i_k}) dx^1 \wedge \cdots \wedge dx^n$ is the set of all local smooth sections

$$\psi : (x^i) \mapsto (x^i, u^\alpha(x^i))$$

of the bundle E such that $u^\alpha(x^i)$ is a solution to the Euler–Lagrange equations $E_\alpha(L) = 0$.

(b) For each $\psi \in S_L$, the tangent space $T_\psi S_L$ at ψ is the set of all vector fields

$$\delta\psi = G^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1} G^\alpha \frac{\partial}{\partial u^\alpha_{i_1}} + D_{i_1} D_{i_2} G^\alpha \frac{\partial}{\partial u^\alpha_{i_1 i_2}} + \ldots \tag{22}$$

on the infinite jet bundle $J^\infty E$ such that $G^\alpha(\psi)$—the pull-back of G^α by the section $\psi \in S_L$—satisfy the Jacobi equations, that is, the linearization of the Euler–Lagrange equations at ψ.

The precise structure of S_L may be quite complicated. For example, a deep result by A. Fisher, J. Marsden, and V. Moncrief states that the space of all (globally hyperbolic) solutions to Einstein's vacuum equations, equipped with a suitable Sobolev topology, is a smooth manifold in the neighborhood of a given solution ${}^{(4)}g_0$ if and only if this metric has no Killing vector fields (Marsden, 1981).

Zuckerman's main result on the existence of a (pre)symplectic form on S_L (Zuckerman, 1987) is the following:

Theorem 4.6. *For any Lagrangian density $\lambda = L dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n \in \Omega^{n,0}(J^\infty E)$, consider the associated differential forms $\eta \in \Omega^{n-1,1}(J^\infty E)$ and $U(\lambda) \in \Omega^{n-1,2}(J^\infty E)$ defined in* (19) *and* (20) *respectively. Then, $U(\lambda)$ is a conserved current for the Euler–Lagrange equations $E_\alpha(L) = 0$. Moreover,*

(a) *Suppose that C is a compact, oriented $(n-1)$-dimensional submanifold of M. Define differential forms θ_C and ω_C on S_L as follows: For any solution $\psi \in S_L$ and any two vectors $\delta_1\psi, \delta_2\psi \in T_\psi S_L$,*

$$\theta_C(\psi) \cdot \delta_1\psi = \int_C \psi^*(i_{\delta_1\psi}\, \eta) \quad \textit{and}$$

$$\omega_C(\psi) \cdot (\delta_1\psi, \delta_2\psi) = \int_C \psi^*(i_{\delta_2\psi} i_{\delta_1\psi}\, U(\lambda)). \tag{23}$$

Then, the one-form θ_C and the two-form ω_C satisfy the equations

$$\omega_C = d\,\theta_C \quad \textit{and} \quad d\,\omega_C = 0. \tag{24}$$

(b) *The two-form ω_C does not depend on the submanifold C.*

We emphasize that this theorem is valid only *on the space S_L*, or "on shell." Thus, Eq. (23) and (24) are valid *only* module the equations of motion $E_\alpha(L) = 0$ and their linearizations.

The two-form ω_C is the (pre)symplectic form on the space of solutions S_L which we were trying to obtain, and (S_L, ω_C) is our version of an evolution space for a general Lagrangian field theory. The important question of when ω_C is in fact symplectic is quite delicate. It is considered in (Crnković and Witten (1987) and Sternberg (1978) for some special cases (Yang–Mills theory and general relativity, and Gel'fand–Dikii formal variational calculus respectively), and in Lee and Wald (1990) in great generality. Some remarks on this issue are also in Zuckerman (1987). In this paper we will study this crucial problem only briefly: details will appear elsewhere. To start with, we need the following definition (Olver 1993; Deligne and Freed, 1999; Zuckerman, 1987):

Definition 4.4. Let $\lambda \in \Omega^{n,0}$ be a Lagrangian density, and let ξ be a vector field on the infinite jet bundle $J^\infty E$ of the form

$$\xi = G^\alpha \frac{\partial}{\partial u^\alpha} + D_{i_1} G^\alpha \frac{\partial}{\partial u^\alpha_{i_1}} + D_{i_1} D_{i_2} G^\alpha \frac{\partial}{\partial u^\alpha_{i_1 i2}} + \cdots.$$

The vector field ξ is a variational symmetry of λ if there exists an $(n, 0)$-form R on $J^\infty E$ such that

$$i_\xi d_V \lambda = d_H R. \tag{25}$$

Variational symmetries are important for two reasons (see Olver, 1993; Zuckerman, 1987 and also Lee and Wald, 1990): first, they are *generalized symmetries* of the Euler–Lagrange equations $E_\alpha(L) = 0$ corresponding to the Lagrangian density $\lambda = L dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$, that is, for any $\psi \in S_L$ the variational symmetry ξ belongs to $T_\psi S_L$; second, Noether's theorem says that there exists a correspondence between these symmetries and conservation laws of the Euler–Lagrange equations $E_\alpha(L) = 0$.

To study the kernel of the presymplectic form ω_C we need to consider a special class of variational symmetries:

Definition 4.5. A local (or gauge) symmetry of a Lagrangian density λ is a one-parameter family of variational symmetries ξ_h of λ—with corresponding $(n, 0)$-forms R_h satisfying Eq. (25)—in which the parameter h is an arbitrary section of a vector bundle $V \to M$ over the space of independent variables, and such that the maps $h \mapsto \xi_h$ and $h \mapsto R_h$ are linear.

This general definition of a local symmetry is due to Zuckerman (1987; Dligne and Freed, 1999). J. Lee and R. M. Wald also arrived to Definition 5 in their deep paper (Lee and Wald, 1990), without using the geometric setting of Zuckerman (1987). These symmetries appear in P. Olver's treatise (Olver, 1993) as well, in the context of Noether's theorem for under-determined Euler–Lagrange systems. Special cases of local symmetries are the gauge transformations of Yang–Mills theory, general relativity and parametrized scalar field theory: these cases are considered in detail in Lee and Wald (1990).

Now, although no complete, rigorous proof has appeared in the literature to this author's knowledge, it seems to be a known fact that the kernel of the two-form ω_C can be characterized in terms of local symmetries:

Theorem 4.7. *The two-form ω_C defined in Eq.* (23) *is degenerate if and only if the Lagrangian density λ admits local symmetries. Moreover, if ω_C is degenerate, its kernel is precisely the linear span of the local symmetries of λ.*

It is expected that a rigorous proof of this result can be obtained by combining the analysis of Barnich *et al.* (1991), Lee and Wald (1990), Torre (1992), and Zuckerman (1987). Assuming Theorem 4 then, the two-form ω_C is indeed a presymplectic form on the space of solutions S_L determined by the Lagrangian density $\lambda = L dx^1 \wedge dx^2 \wedge \cdots \wedge dx^n$. By analogy with the theory of Section 2 we construct the *covariant phase space for the (field) theory described by* λ by applying Souriau's reduction to the evolution space (S_L, ω_C):

Definition 4.6. Let $\mathcal{G}$ be the span of all the local symmetries admitted by the Lagrangian density $\lambda = L dx^1 \wedge dx^2 \wedge \ldots \wedge dx^n$. The space

$$\mathcal{M} = S_L/\ker \omega_C = S_L/\mathcal{G}$$

equipped with the symplectic form $\widetilde{\omega_C}$ determined by the equation $\pi^* \widetilde{\omega_C} = \omega_C$, in which $\pi : S_L \to \mathcal{M}$ is the canonical projection, is called the covariant phase space of the theory determined by λ.

We remark that this definition is only formal, as we are silent with respect to the precise structure of $\mathcal{M}$. Of course $(\mathcal{M}, \widetilde{\omega_C})$ corresponds to the space of motions U_M of Section 2 (see Definition 1 and Theorem 2) but in that section we used Frobenius' theorem to construct U_M, and it was of importance for us to have a general criterium guaranteeing that U_M can be given a manifold structure. In the case of a general Lagrangian field theory, we do not know a priori whether $\mathcal{G}$ will determine a foliation on S_L and, assuming that such a foliation exists, neither do we known whether the space of leaves will admit a manifold structure: it appears that all this must be checked on a case by case basis. We refer the reader to Zuckerman (1987) where G. Zuckerman presents some important examples for which bona fide covariant phase spaces can be built.

Finally, we would like to mention that we have not considered in this paper the relevance of covariant phase spaces for quantization of classical systems. For this, the reader is referred to Aldaya *et al.* (1992a,b,c). and also to Zuckerman's (1987) paper.

5. AN EXAMPLE

In this final section we consider the simple example of a Lagrangian density of the form (Torre, 1992; Woodhouse, 1992)

$$\lambda = L(x^i, u^\alpha, u_i^\alpha)\, dx^1 \wedge \cdots \wedge dx^n.$$

We set $\nu = dx^1 \wedge \cdots \wedge dx^n$, and $\nu_i = (-1)^i\, dx^1 \wedge \cdots \wedge dx^{i-1} \wedge dx^{i+1} \wedge \cdots \wedge dx^n$. We then find

$$d_V \lambda = \left(\frac{\partial L}{\partial u^\alpha}\, \theta^\alpha + \frac{\partial L}{\partial u_i^\alpha}\, \theta_i^\alpha \right) \wedge \nu = E_\alpha(L)\, \theta^\alpha \wedge \nu$$
$$+ \left[\frac{\partial L}{\partial u_i^\alpha}\, \theta_i^\alpha + D_i \left(\frac{\partial L}{\partial u_i^\alpha} \right) \theta^\alpha \right] \wedge \nu$$

and, on the other hand, we easily compute

$$d_H \left(\frac{\partial L}{\partial u_i^\alpha}\, \theta^\alpha \wedge \nu_i \right) = \left[\frac{\partial L}{\partial u_i^\alpha}\, \theta_i^\alpha + D_i \left(\frac{\partial L}{\partial u_i^\alpha} \right) \theta^\alpha \right] \wedge \nu.$$

Thus, $d_V \lambda$ can be written as $d_V \lambda = E_\alpha(L)\theta^\alpha \wedge \nu + d_H \eta$ in which

$$\eta = \frac{\partial L}{\partial u_i^\alpha}\, \theta^\alpha \wedge \nu_i. \tag{26}$$

The "presymplectic potential" θ_C defined in Eq. (23) now reads

$$\theta_C(\psi) \cdot \delta\psi = \int_C \psi^*(i_{\delta\psi}\eta) = \int_C \frac{\partial L}{\partial u_i^\alpha}\, G^\alpha\, \nu_i, \tag{27}$$

in which $\delta\psi$ is given by (22). This formula, found here from general principles, coincides with the ones Woodhouse (1992, p.132) and Crnković (1987, 1988) found by formal manipulations. The corresponding (pre)symplectic form ω_C given by

$$\omega_C(\psi) \cdot (\delta_1\psi, \delta_2\psi) = \int_C \psi^* \left(i_{\delta_2\psi} i_{\delta_1\psi}\, U(\lambda) \right) = \int_C \psi^* \left(i_{\delta_2\psi} i_{\delta_1\psi}\, d_V \eta \right),$$

in which $\delta_1\psi$ and $\delta_2\psi$ are vectors in $T_\psi S_L$ as in (22), becomes

$$\omega_C(\psi)\cdot(\delta_1\psi,\delta_2\psi) =$$
$$\int_C \left(\frac{\partial^2 L}{\partial u^\beta \partial u_i^\alpha} \left[G_1^\beta G_2^\alpha - G_2^\beta G_1^\alpha\right] + \frac{\partial^2 L}{\partial u_j^\beta \partial u_i^\alpha}\left[\left(D_j G_1^\beta\right)G_2^\alpha - \left(D_j G_2^\beta\right)G_1^\alpha\right]\right)\nu_i, \tag{28}$$

coinciding again with Woodhouse's (1992) formulae.

We apply these computations to the homogeneous Monge–Ampere equation

$$\frac{\partial^2 u}{\partial t^2}\,\frac{\partial^2 u}{\partial x^2} = \left(\frac{\partial^2 u}{\partial t \partial x}\right)^2. \tag{29}$$

The Hamiltonian structure of this equation has been studied with great care by Nutku (1996, 2001). Nutku uses mainly Dirac's theory of constraints, but he also discusses Eq. (29) briefly from the point of view considered here. The Lagrangian density associated to (29) is (Nutku, 1996, 2001)

$$\lambda = \left(l(u_x, q)q_t - \frac{\partial l(u_x,q)}{\partial u_x}(u_t - q)q_x \right) dx \wedge dt, \tag{30}$$

in which $l(u_x, q)$ is an arbitrary smooth function depending on u_x and q such that $l_{u_x u_x} \neq 0$. The Euler–Lagrange equations corresponding to (30) are

$$-u_{xx}qt + 2q_x u_{xt} - q_x^2 = 0 \quad \text{and} \quad u_{xx}u_t = u_{xx}q,$$

that is,

$$u_t = q \quad \text{and} \quad q_t = \frac{1}{u_{xx}}\, q_x^2, \tag{31}$$

which is of course equivalent to (29). If we set

$$\delta\psi = G^u \frac{\partial}{\partial u} + G^q \frac{\partial}{\partial q} + D_x G^u \frac{\partial}{\partial u_x} + D_t G^u \frac{\partial}{\partial u_t} + D_x G^q \frac{\partial}{\partial q_x} + D_t G^q \frac{\partial}{\partial q_t} + \dots, \tag{32}$$

Equation (27) for the (pre)symplectic potential θ_C becomes

$$\theta_C(\psi)\cdot\delta\psi = \int_C -l_{u_x}\, q_t\, G^u\, dt + (l\, G^q - l_{u_x}\, q_x\, G^u)\, dx$$

whenever $\psi = (u(x,t), q(x,t))$ is a solution to Eq. (31), while the (pre) symplectic form (28) reads

$$\omega_C(\psi)\cdot(\delta_1\psi,\delta_2\psi) =$$
$$\int_C \{ -l_{u_x q} q_x\left[G_1^q G_2^u - G_2^q G_1^u\right] - l_{u_x u_x} q_x \left[\left(D_x G_1^u\right)G_2^u - \left(D_x G_2^u\right)G_1^u\right]$$

$$
\begin{aligned}
&- l_{u_x}\big[\big(D_x G_1^q\big)G_2^u - \big(D_x G_2^q\big)G_1^u\big] + l_{u_x}\big[\big(D_x G_1^u\big)G_2^q - \big(D_x G_2^u\big)G_1^q\big]\,\big\}\,dx \\
&- \big\{l_{u_x q} q_t\big[G_1^q G_2^u - G_2^q G_1^u\big] + l_{u_x u_x} q_t\big[(D_x G_1^u)G_2^u - \big(D_x G_2^u\big)G_1^u\big] \\
&+ l_{u_x}\big[\big(D_t G_1^q\big)G_2^u - \big(D_t G_2^q\big)G_1^u\big]\big\}\,dt \qquad (33)
\end{aligned}
$$

in which $\delta_1\psi$ and $\delta_2\psi$ are vectors in $T_\psi S_L$ as in (32), and the equations $u_t = q$ and $D_t G^u = G^q$ have been used to simplify Eq. (28) for ω_C. Eq. (33) can be simplified further using integration by parts and the equations $u_t = q$ and $D_t G^u = G^q$. A straightforward computation yields

$$
\begin{aligned}
&\omega_C(\psi)\cdot(\delta_1\psi, \delta_2\psi) = \\
&\int_C l_{u_x u_x}\left(u_{xx}\big[G_1^q G_2^u - G_2^q G_1^u\big] - q_x\big[\big(D_x G_1^u\big)G_2^u - \big(D_x G_2^u\big)G_1^u\big]\right)dx \\
&\quad + l_{u_x u_x}\left(q_x\big[G_1^q G_2^u - G_2^q G_1^u\big] - \frac{q_x^2}{u_{xx}}\big[\big(D_x G_1^u\big)G_2^u - \big(D_x G_2^u\big)G_1^u\big]\right)dt. \qquad (34)
\end{aligned}
$$

It is possible to check directly (that is, without using Theorem 4.7.) that the form ω_C given by (34) is nondegenerate. We can thus conclude that (S_L, ω_C), with L determined by (30), is the covariant phase space for the homogeneous Monge–Ampere equation.

We would like to finish by pointing out a few problems to be considered elsewhere. First, as already stressed by Nutku (1996, 2001), it is interesting to note that Lagrangian methods do not much appear in integrable systems, the exception being the intriguing paper by Sternberg (1978). We wonder if the analysis of Sternberg can be generalized to other integrable hierarchies. Second, we also wonder if Zuckerman's (1987) approach can be related to the Hamiltonian operators of Olver (1989, 1993). Third, with respect to the Monge–Ampere example, Nutku (1996, 2001) points out that Eq. (29) possesses a multi-hamiltonian structure (Olver, 1993). One wonders how this rich structure reflects itself at the phase space level considered here.

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